Two-dimensional electrodynamics and the global structure of space

1986 J. Phys. A: Math. Gen. 19 2099

(http://iopscience.iop.org/0305-4470/19/11/018)

J. Phys. A: Math. Gen. 19 (1986) 2099-2104. Printed in Great Britain

Two-dimensional electrodynamics and the global structure of space

G McHale and G A Jaroszkiewicz

Department of Mathematics, University of Nottingham, University Park, Nottingham NG7 2RD, UK

Received 19 March 1985, in final form 7 November 1985

Abstract. Two-dimensional electrodynamics based on the spacetime $S^1 \otimes \mathbb{R}$ is formulated using Fourier series and Dirac's constraint theory. The role of the background electric field F as a dynamical variable is established. We show that the background field is unstable against a class of charged pair creation processes involving the global topology of S^1 provided $|F| > \frac{1}{2}e$, where e is the unit of electric charge. Our results do not require any discussion of the Coulomb Green function or of the role of charges on spatial boundaries.

1. Introduction

The motivation for this paper arises from our studies of the Schwinger model (Schwinger 1962, Lowenstein and Sweica 1971) (QED in 1+1 dimensions with massless fermions), which we hoped would give us insight into the solvable aspects of QED in 1+3 dimensions. The Schwinger model is one of a few completely soluble quantum field theories, and its extension to the massive fermion case has been investigated in some detail (Coleman 1976, Coleman *et al* 1975).

In two spacetime dimensions, the electric field F_{01} may be written as

$$F_{01}(x, t) = \int dy\, K(x, y) j_0(y, t) + F$$

where the Coulomb Green function $K(x, y)$ satisfies the equation $(\partial/\partial x)K(x, y) = -\delta(x-y)$, $j_0(y, t)$ is the charge density and F is an arbitrary constant. Coleman (1976) referred to F as the background electric field, and argued that it would be stable against spontaneous charged pair creation provided $|F| \leq \frac{1}{2}e$, where e is the unit of electric charge. The argument involves a solution to the classical Euler-Lagrange equations for the electromagnetic fields in the presence of external charges, after first gauging the spatial component of the vector potential to zero (the 'axial' gauge). This phenomenon has stimulated discussions on the role of inequivalent vacua in gauge theories. Coleman's spacetime is $\mathbb{R} \otimes \mathbb{R}$ with the usual Minkowski metric.

Subsequently, Tyburski (1981) re-examined Coleman's argument, truncating spacetime to $[-L, L] \otimes \mathbb{R}$ and imposing periodic boundary conditions. He argued that the background field is a dynamical variable and, regardless of its magnitude, is stable against spontaneous pair creation. Furthermore, the axial gauge cannot be imposed in his model. Both Coleman and Tyburski discussed the relationship of the background field to charges at spatial boundaries, and the arguments of each depend on the Coulomb Green functions used.

0305-4470/86/112099+06\$02.50

We reformulate Tyburski's model on a spacetime with the topology of $S^1\otimes\mathbb{R}$, with the requirement that all functions be single-valued. This allows us to avoid discussing charges on spatial boundaries in a natural way. After specifying the Lagrangian for electromagnetic potentials in the presence of external sources, we follow Tyburski (1981) and employ a Fourier series expansion of all fields and currents. Then, by systematically applying Dirac's constraint theory (Dirac 1964) we show that Coleman's results can be obtained for $S^1\otimes\mathbb{R}$ without a discussion of the Coulomb Green function. This result is gauge independent, and the same approach proves that, classically, states of non-zero net charge are inconsistent with the dynamics. This last result is intimately linked to the topology of our spacetime.

2. General formalism

On $S^1\otimes\mathbb{R}$ the electromagnetic potentials $A^\mu=(\varphi, A)$ may be expanded as a Fourier series:

$$\begin{aligned} A(x,t)&=\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_nC_n+b_nS_n)\\ \varphi(x,t)&=\tfrac{1}{2}g_0+\sum_{n=1}^{\infty}(g_nC_n+h_nS_n)\end{aligned}\tag{2.1}$$

where the coefficients $a_0, a_1, \ldots, g_0, g_1, \ldots$, are time dependent and $C_n=\cos(n\pi x/L)$, $S_m=\sin(m\pi x/L)$ are elements of a complete orthogonal set on S^1.

The chosen Lagrangian is

$$L[\rho, j]=\int_{-L}^{L}(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+j_\mu A^\mu)\,\mathrm{d}x \tag{2.2}$$

where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ are components of the Faraday tensor and $j^\mu=(\rho, j)$ is the classical external charge density current. Of course, our choice of spacetime restricts the class of coordinate transformations which leave the Lagrangian (2.2) invariant. We assume that in the limit $L\to\infty$ we recover the usual relativistic framework. The same assumption is made in Tyburski's approach. With the expansions (2.1) the Lagrangian (2.2) becomes

$$\begin{aligned} L[\rho,j]&=\tfrac{1}{4}L\dot a_0^2-\tfrac{1}{2}a_0A_0+\tfrac{1}{2}g_0G_0+\tfrac{1}{2}L\sum_{n=1}^{\infty}(\dot a_n^2+\dot b_n^2)\\ &\quad+\sum_{n=1}^{\infty}n\pi(\dot a_nh_n-\dot b_ng_n)+\tfrac{1}{2}L^{-1}\sum_{n=1}^{\infty}n^2\pi^2(g_n^2+h_n^2)\\ &\quad+\sum_{n=1}^{\infty}(g_nG_n+h_nH_n-a_nA_n-b_nB_n)\end{aligned}\tag{2.3}$$

where $A_n=(j, C_n)$, $B_n=(j, S_n)$, $G_n=(\rho, C_n)$, $H_n=(\rho, S_n)$ with $(u, v)=\int_{-L}^{L}u(x)v(x)\,\mathrm{d}x$.

We regard the coefficients $a_0, a_1, \ldots, g_0, \ldots, h_1, \ldots$, as the dynamical degrees of freedom and apply Dirac's formalism for constrained systems (Dirac 1964) to the Lagrangian (2.3). For each degree of freedom q the canonical conjugate momentum $\bar q\equiv\partial L[\rho,j]/\partial\dot q$ has canonical Poisson bracket (PB),$\{\bar q, q\}_{\mathrm{PB}}=-1$ with all other brackets zero. The strong equalities

$$\bar a_0=\tfrac{1}{2}L\dot a_0$$
$$\bar a_n=L\dot a_n+n\pi h_n \qquad \bar b_n=L\dot b_n-n\pi g_n \qquad n\geqslant 1$$

are used subsequently to constrain the possible gauge transformations. We find the primary constraints

$$\bar{g}_n \approx 0 \qquad n \geqslant 0$$
$$\bar{h}_n \approx 0 \qquad n \geqslant 1 \tag{2.4}$$

and, subsequently, the secondary constraints

$$\begin{aligned} -(n\pi/L)\bar{b}_n + G_n \approx 0 \qquad n \geqslant 1 \\ (n\pi/L)\bar{a}_n + H_n \approx 0 \qquad n \geqslant 1. \end{aligned} \tag{2.5}$$

Clearly, all momenta are constrained except $\bar{a}_0$; we identify this momentum as proportional to the background field $P(t)$ of Tyburski (1981).

Significantly, the consistency condition $G_0 = 0$, i.e.

$$\int_{-L}^{L} \rho(x, t)\, \mathrm{d}x = 0$$

emerges directly from the formalism. This means we have no choice but to work with states of total charge zero on S^1. This result may be viewed in two ways:

(i) as a consistency condition obtained from Dirac's constraint theory applied to the Lagrangian (2.3), with no reference to global topology, or

(ii) as a consequence of the topology of S^1. An isolated electric charge e radiates a constant electric field in each direction. These fields must meet somewhere in S^1, and be absorbed by a sink charge $-e$ in order to maintain a single-valued field.

On the surface of constraint in phase space the Hamiltonian becomes

$$H[\rho, j] \approx L^{-1}\bar{a}_0^2 + \tfrac{1}{2}a_0 A_0 + \tfrac{1}{2}L \sum_{n=1}^{\infty} (n\pi)^{-2}(G_n^2 + H_n^2) + \sum_{n=1}^{\infty} (a_n A_n + b_n B_n). \tag{2.6}$$

The temporal stability of the constraints (2.4) and (2.5) imposes the following conditions on the current:

$$n\pi B_n = -L\dot{G}_n$$
$$n\pi A_n = L\dot{H}_n \qquad n \geqslant 1$$

which corresponds to the continuity equation

$$\dot{\rho} + \partial_x j = 0.$$

All constraints are first-class and generate the gauge transformation

$$A^\mu \to A^\mu - \partial^\mu \Lambda$$

where Λ is periodic on $[-L, L]$ (Tyburski 1981), or in our terms single-valued on S^1. Clearly, a_0 is gauge invariant, and consequently the axial gauge ($A = 0$) cannot be used in our model, a result also found in Tyburski's formulation. Although the equations of motion are found to be gauge invariant, we note that the Hamiltonian on the surface of constraints (2.6) is not. This is because our original Lagrangian (2.2) neglects the constituent fields of the current, which would provide compensating gauge terms. However, the energy changes we calculate below are all gauge invariant.

In § 4 we make use of the equation of motion for $\bar{a}_0$, given by

$$\dot{\bar{a}}_0 = -\tfrac{1}{2}A_0 = -\tfrac{1}{2}\int_{-L}^{L} \mathrm{d}x\, j(x, t). \tag{2.7}$$

This shows that the background field is not independent of the charge current, a point overlooked by Tyburski.

3. Static charge pair

If we consider the static charge case then $j=0$ and the energy $E(\rho, 0)$ of the system is given by

$$E(\rho, 0)=L^{-1}\bar{a}_0^2+\tfrac{1}{2}L\sum_{n=1}^{\infty}(n\pi)^{-2}(G_n^2+H_n^2)$$

which is gauge invariant. The only dynamical variable in this case is $\bar{a}_0$, which is time independent because $j=0$ in (2.7). We cannot gauge this variable away, because it has zero Poisson bracket with all the first-class constraints, and we identify it as proportional to the background electric field of Coleman and Tyburski.

Tyburski's motivation for truncating space was to provide a finite procedure for placing charges on boundaries and then taking the limit $L\to\infty$. Care must be taken in such cases. To illustrate, we consider a state containing a static charge density

$$\rho(x)=e[\delta^L(x-r)-\delta^L(x+r)] \qquad 0\leqslant r<L$$

where $\delta^L(x)$ is the periodic delta function on S^1 defined by its Fourier series

$$\delta^L(x)=\tfrac{1}{2}L^{-1}+L^{-1}\sum_{n=1}^{\infty}\cos\left(\frac{n\pi x}{L}\right).$$

Then

$$G_n=0$$
$$H_n=2e\sin(n\pi r/L) \qquad n\geqslant 0$$

giving

$$E(r)=L^{-1}\bar{a}_0^2+e^2r(1-r/L) \tag{3.1}$$

for the energy of the state. This energy is invariant under the substitution $r\to L-r$, which reflects the global topology of S^1.

If we calculate the energy difference $\Delta E=E(r)-E(0)$ we find

$$\Delta E=e^2r(1-r/L) \tag{3.2}$$

which is never negative. This argument was used by Tyburski to substantiate his claim that the introduction of charged pairs could not decrease the energy of the system, in contradiction with Coleman's result for $\mathbb{R}\otimes\mathbb{R}$. The result holds if we take the limit $r\to L$ in (3.2), which corresponds to placing charges on the boundaries of Tyburski's space $[-L, L]$.

However, this argument is incorrect because it ignores the coupling (2.7) of the background field to the charge currents, which necessarily arise whenever we move, create, or annihilate charges. In the next section we present a vacuum fluctuation process which has no affect other than to change the background field.

4. Vacuum fluctuations

We consider a vacuum fluctuation process described by the following charge and current densities

$$\rho(x, t)=e[\delta^L(x-ut)-\delta^L(x+ut)][\theta(t)-\theta(t-L/u)]$$
$$j(x, t)=eu[\delta^L(x-ut)+\delta^L(x+ut)][\theta(t)-\theta(t-L/u)]$$

where $\theta(t)$ is the unit step function. These densities model a vacuum fluctuation process which starts with no charges in the system for negative times. At time $t=0$ a charge pair is created at the origin. Subsequently, these charges separate with equal and opposite velocities, finally annihilating at the antepodal point $x=\pm L$ at time $t=L/u$. For times $t>L/u$ the system has apparently reverted to its initial charge-free state. We now show this is not the case.

For negative times the initial energy of the system E_i depends only on the initial background field $F=L^{-1}\bar{a}_0(-\infty)$, given by

$$E_i = LF^2.$$

During the vacuum fluctuation, $\bar{a}_0(t)$ changes according to (2.7), and for $t>L/u$ we find $\bar{a}_0(\infty)=\bar{a}_0(-\infty)-eL$, giving

$$E_f = L(F-e)^2$$

for the energy of the system after the vacuum fluctuation process has ended. The change in energy produced by the vacuum fluctuation is therefore

$$\Delta E = E_f - E_i = Le(e-2F).$$

This can be negative if $|F|>\frac{1}{2}e$.

Physically, the above process corresponds to the creation of a closed electric flux string around S^1, adding or subtracting a field strength e to the original background field. This is the $S^1\otimes\mathbb{R}$ analogue of Coleman's $\mathbb{R}\otimes\mathbb{R}$ result. Of course, the appearance in this classical theory of apparently quantised units of electric flux depends on the assumption that classical electric charges occur in multiples of the fundamental charge e only.

It could be argued that the speed of light is a natural upper bound to the pair separation speed u in the above process, and that a value of L of cosmological proportions would imply a long time scale for the above process to be completed. This argument can be circumvented by considering the spontaneous creation of many charged pairs distributed uniformly around S^1, specified by the charge and current densities:

$$\rho(x,t) = e\sum_{n=1-N}^{N}[\delta^L(x-2an-ut)-\delta^L(x-2an+ut)][\theta(t)-\theta(t-a/u)]$$

$$j(x,t) = eu\sum_{n=1-N}^{N}[\delta^L(x-2an-ut)+\delta^L(x-2an+ut)][\theta(t)-\theta(t-a/u)]$$

where $L=2aN$. This process corresponds to the creation of $2N$ charged pairs, whose equal and opposite members separate and travel for a period $L/2Nu$ until they meet and annihilate opposite members of adjacent pairs. The net result is the addition of a closed electric flux loop of strength e to the background field, as before. In this case the time of duration of the process may be made arbitrarily small by taking N large enough, keeping u below the speed of light.

5. Summary

We have examined the stability of the background electric field F in a classical model of two-dimensional electrodynamics on the spacetime $S^1\otimes\mathbb{R}$. We have shown that F

is unstable against a class of vacuum pair creation processes provided $|F| > \frac{1}{2}e$. This result supports Coleman's arguments concerning the origin of inequivalent or θ vacua in the Schwinger model.

The phenomenon responsible for this instability is the spontaneous creation of closed electric flux strings of field strength e. The result is determined by the dynamical nature of the background field and the topology of our spacetime, is gauge invariant, and requires no discussion of the Coulomb Green function or of charges on spatial boundaries. Furthermore, we have shown that the axial gauge cannot be used in the model and that states of non-zero net electric charge cannot exist.

Acknowledgments

We are indebted to one of the referees for pointing out an important oversight in the first version of this paper. One of us (GM) acknowledges the support of the Science and Engineering Research Council.

References

Coleman S 1976 *Ann. Phys., NY* **101** 239-67
Coleman S, Jackiw R and Susskind L 1975 *Ann. Phys., NY* **93** 267-73
Dirac P A M 1964 *Lectures on Quantum Mechanics* (New York: Belfer Graduate School of Science)
Lowenstein J H and Sweica J A 1971 *Ann. Phys., NY* **68** 172-95
Schwinger J 1962 *Phys. Rev.* **128** 2425-9
Tyburski L 1981 *Phys. Lett.* **99B** 218-20